Last Time: Change of Basis!

Represent a linear map $L: V \to W$ via many matrices...

Special Cases: If $V = W$ and $L = id$.

$\text{Rep}_{B,D}(id)$ is the matrix representing the change of basis B to D.

$$
\begin{tikzcd}
V_B \arrow[r, "\text{Rep}_{B,D}(L)"] \arrow[d, shift right, "\text{Rep}_{B,B'}(id)"'] & W_D \arrow[d, "\text{Rep}_{D,D'}(id)"] \\
V_{B'} \arrow[r, "\text{Rep}_{B',D'}(L)"'] \arrow[u, shift right, bend right, "\text{Rep}_{B',B}(id)"'] & W_{D'}
\end{tikzcd}
$$

$$\text{Rep}_{B',D'}(L) = R_{D,D'}(id) \cdot \text{Rep}_{B,D}(L) \cdot \text{Rep}_{B',B}(id)$$

WHY?: Some bases make for really simple representations of your linear map...

Remark: Some "nice" linear operators can be represented by diagonal matrices...

Ex: Consider the spaces $V = \mathcal{P}_2(\mathbb{R})$ and $W = \mathcal{M}_{2\times 2}(\mathbb{R})$.

$$B = \{1,\ 1+x,\ 1+x^2\},\quad B' = \{1,\ x,\ x^2\} \subseteq V$$

$$D = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightsquigarrow \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$D' = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Ⓐ Build $\underline{\mathrm{Rep}_{B,B'}}(id_V)$ and $\mathrm{Rep}_{D,D'}(id_W)$.

$$[B' \mid B] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \underline{\mathrm{Rep}_{B,B'}(id_V) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$[D' \mid D] \rightsquigarrow \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\therefore \mathrm{Rep}_{D,D'} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Ⓑ Suppose $L : V \to W$ is represented by

$$\mathrm{Rep}_{B,D}(L) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

What is $\mathrm{Rep}_{B',D'}(L)$?

$$\begin{tikzcd} V_B \arrow[r, "\mathrm{Rep}_{B,D}(L)"] \arrow[d, "\mathrm{Rep}_{B,B'}(id)"'] & W_D \arrow[d, "\mathrm{Rep}_{D,D'}(id)"] \\ V_{B'} \arrow[r, "\mathrm{Rep}_{B',D'}(L)"'] & W_{D'} \end{tikzcd}$$

$$\therefore \ \underline{\text{Rep}_{B',D'}(L)} = \underline{R_{D,D'}(id)} \cdot \text{Rep}_{B,D}(L) \cdot \text{Rep}_{B',B}(id)$$

Now we compute $\text{Rep}_{B',B}(id) = \text{Rep}_{B,B'}(id)^{-1}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$\therefore \ \text{Rep}_{B',B}(id) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so finally:

$$\text{Rep}_{B',D'}(L) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$$

$\blacksquare$

Eigenvectors and Eigenvalues

Goal: Understand when a matrix can be diagonalized...

↳ On hold... we'll build up to this :)

Defn: Let $L : V \to V$ be a linear operator.

① An eigenvector of L is an element $v \in V$ such that $L(v) = \lambda v$ for some scalar λ.

② The *eigenvalue* of eigenvector $v \in V$ for L is the scalar λ with $L(v) = \lambda v$.

More succinctly: An *eigenvector* of L w/ *eigenvalue* λ is a vector $v \in V$ with $L(v) = \lambda v$. ☆

NB: "eigen" means (roughly) "same" in German.

Ex: Consider the transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ given by $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x \\ 5y \\ 0 \end{pmatrix}$. Note that

$L(e_1) = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 3e_1$ so e_1 is an eigenvector of L with eigenvalue 3.

$L(e_2) = 5e_2$ so e_2 is an eigenvector of L w/ eigenvalue 5.

$L(e_3) = \vec{0}$ So e_3 is a eigenvector w/ eigenvalue 0...

$L(\vec{0}) = \vec{0} = \lambda \vec{0}$ for all $\lambda \in \mathbb{R}$...

But for technical reasons, we do NOT call $\vec{0}$ an eigenvector...

Remark: v is an eigenvector w/ eigenvalue 0 if and only if $v \in \ker(L)$.

↳ Exercise: prove it!

Prop: If v, w are eigenvectors of L w/ eigenvalue λ, then ① av also has eigenvalue λ.
② $v+w$ also has eigenvalue λ.

Q: How do I compute eigenvalues and eigenvectors?

Note: $L(v) = \lambda v$ if L is represented by

$\operatorname{Rep}_{B,B}(L) = M$, then we're asking for:

$$Mu = \lambda u = \lambda I_n u$$

so $\quad Mu - \lambda I_n u = 0$

i.e. $\underbrace{(M - \lambda I_n)}\, u = 0$

So this transformation has u in its kernel...

Thus $\underline{\det(M - \boxed{\lambda} I_n)} = 0 \ldots$

Def^n: The characteristic polynomial of matrix M (or more generally the operator associated to M) is the polynomial $p_M(\lambda) := \det(M - \lambda I)$.

Point: Every eigenvalue of M is a root of the characteristic polynomial $p_M(\lambda)$.

Ex: Compute $p_M(\lambda)$ for $M = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol: $p_M(\lambda) = \det(M - \lambda I)$

$$= \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & 1 & -\lambda \end{bmatrix} = (1-\lambda)\det\begin{bmatrix} 1-\lambda & -1 \\ 1 & -\lambda \end{bmatrix} - \det\begin{bmatrix} 0 & 1 \\ 1 & -\lambda \end{bmatrix} + 0$$

$$= (1-\lambda)\big(-\lambda(1-\lambda) + 1\big) - (-1)$$

$$= (1-\lambda)(1 - \lambda + \lambda^2) + 1$$

$$= (1-\lambda+\lambda^2) - \lambda(1-\lambda+\lambda^2) + 1$$

$$= 1-\lambda+\lambda^2-\lambda+\lambda^2-\lambda^3+1$$

$$= -\lambda^3+2\lambda^2-2\lambda+2$$ $\boxed{}$

<u>Ex:</u> Consider $L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5x \\ -y \\ 0 \end{pmatrix}$. This transformation has matrix $M = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ wrt $\mathcal{E}_3$. We

Compute $p_M(\lambda) = \det(M - \lambda I)$

$$= \det \begin{bmatrix} 5-\lambda & 0 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = (5-\lambda)(-1-\lambda)(-\lambda)$$

$$= \lambda(\lambda+1)(5-\lambda).$$

which has roots $\lambda = 0$, $\lambda = -1$, and $\lambda = 5$. $\boxed{}$